

ALGEBRAIC REFLEXIVITY OF THE SET OF n -ISOMETRIES ON $C(X, E)$

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ABSTRACT. We prove that if the group of isometries of $C(X, E)$ is algebraically reflexive, then the group of n -isometries is also algebraically reflexive. Here, X is a compact Hausdorff space and E is a Banach space. As a corollary to this, we establish the algebraic reflexivity of the set of generalized bi-circular projections on $C(X, E)$. This answers a question raised in [1].

1. INTRODUCTION

Let X be a compact Hausdorff space and E be a Banach space. We denote by $\mathcal{G}(E)$ the group of all surjective isometries of E . Suppose that E has trivial centralizer. Then any isometry T of $C(X, E)$ is of the form $Tf(x) = \tau(x)(f(\phi(x)))$ for $x \in X$ and $f \in C(X, E)$, where $\tau : X \rightarrow \mathcal{G}(E)$ continuous in strong operator topology and ϕ is a homeomorphism of X onto itself. Let $\mathcal{G}^n(C(X, E)) = \{T \in \mathcal{G}(C(X, E)) : T^n = I\}$. Any $T \in \mathcal{G}^n(C(X, E))$ is called an n -isometry. It can be easily proved that $T \in \mathcal{G}^n(C(X, E))$ if and only if there exist a homeomorphism ϕ of X such that $\phi^n(x) = x$ for all $x \in X$, a map $\tau : X \rightarrow \mathcal{G}(E)$ satisfying $\tau(x) \circ \tau(\phi(x)) \circ \dots \circ \tau(\phi^{n-1}(x)) = I$, where I denotes the identity map and T is given by $Tf(x) = \tau(x)(f(\phi(x)))$, for all $x \in X$ and $f \in C(X, E)$.

Let $B(E)$ be the set of all bounded linear operators on E and S is any subset of $B(E)$. The Algebraic closure \overline{S}^a of S is defined as follow: $T \in \overline{S}^a$ if for every $e \in E$ there exists $T_e \in S$ such that $T(e) = T_e(e)$. S is said to be algebraically reflexive if $S = \overline{S}^a$. Let \mathbb{T} denotes the unit circle. A linear projection $P : E \rightarrow E$ is said to be a generalized bi-circular projection (GBP for short) if for some $\lambda \in \mathbb{T}$, $P + \lambda(I - P)$ is an isometry.

It was proved by Dutta and Rao, see [1], that for a compact Hausdorff space X , if $\mathcal{G}(C(X))$ is algebraically reflexive then $\mathcal{G}^2(C(X))$ is also algebraically reflexive. In this note we prove this result for vector valued continuous functions and for any $n \geq 2$. We also answer a question raised by them about the algebraic reflexivity of the set of GBP's of $C(X, E)$.

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2. MAIN RESULTS

Theorem 2.1. *Let X be a compact Hausdorff space and E be a Banach space. If $\mathcal{G}(C(X, E))$ is algebraically reflexive, then $\mathcal{G}^n(C(X, E))$ is also algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}^n(C(X, E))}^a$. Then, for each $f \in C(X, E)$ we have $Tf(x) = \tau_f(x)(f(\phi_f(x)))$ where $\tau_f : X \rightarrow \mathcal{G}(E)$ is continuous in s.o.t satisfying $\tau_f(x) \circ \tau_f(\phi(x)) \circ \dots \circ \tau_f(\phi^{n-1}(x)) = I$ and ϕ_f is a homeomorphism of X such that $\phi_f^n(x) = x$ for all $x \in X$. By the algebraic reflexivity of $\mathcal{G}(C(X, E))$, T is an isometry and hence $Tf(x) = \tau(x)(f(\phi(x)))$ where $\tau(x) \in \mathcal{G}(E)$ for each x and ϕ is a homeomorphism of X . To show that $T^n = I$ we need to show that $\phi^n(x) = x$ and $\tau(x) \circ \tau(\phi(x)) \circ \dots \circ \tau(\phi^{n-1}(x)) = I$.

Firstly suppose that f is a rank one tensor product, i.e., $f = h \otimes e$, where h is a strictly positive function $C(X)$ and $e \in E$. Since $\tau(x)(f(\phi(x))) = \tau_f(x)(f(\phi_f(x)))$, we get $\tau(x)(h(\phi(x))e) = \tau_f(x)(h(\phi_f(x))e)$. As $\tau(x)$ and $\tau_f(x)$ are both isometries we have $h(\phi(x)) = h(\phi_f(x))$. Therefore, $\tau(x) = \tau_f(x)$ for all $x \in X$.

Now consider any point $x \in X$.

Case(i): $x = \phi(x)$.

Then $\phi^n(x) = \phi(\phi(\dots(\phi(x))\dots))(n \text{ times}) = x$. Choose $h \in C(X)$ such that $0 < h \leq 1$ and $h^{-1}(1) = \{x\}$. For $f = h \otimes e$, $e \in E$, evaluating Tf at x we get $Tf(x) = \tau(x)(f(\phi(x))) = \tau(x)(h(x)e) = \tau(x)(e) = \tau_f(x)(f(\phi_f(x))) = \tau_f(x)(h(\phi_f(x))e)$. As $\tau(x)$ and $\tau_f(x)$ are isometries, by the choice of h we have $\phi_f(x) = x$. Now, $I = \tau_f(x) \circ \tau_f(\phi_f(x)) \circ \dots \circ \tau_f(\phi_f^{n-1}(x)) = \tau_f(x) \circ \tau_f(x) \circ \dots \circ \tau_f(x) = \tau(x) \circ \tau(x) \circ \dots \circ \tau(x)$.

Case(ii): $\phi(x) \neq x$, $\phi^m(x) = x$, m divides n and $\phi^s(x) \neq x \forall s$ such that $1 \leq s < m$.

As m divides n , $n = mq$ for some positive integer q . Therefore, $\phi^n(x) = \phi^{mq}(x) = \phi^m(\phi^m(\dots(\phi^m(x))\dots))(q \text{ times}) = x$. Now, choose $h \in C(X)$ such that $1 \leq h \leq m$ and $h^{-1}(1) = \{x\}$, $h^{-1}(2) = \{\phi(x)\}, \dots, h^{-1}(m) = \{\phi^{m-1}(x)\}$. Let $f = h \otimes e$, for any $e \in E$. Evaluating Tf at x we get $Tf(x) = \tau(x)(f(\phi(x))) = \tau(x)(h(\phi(x))e) = \tau(x)(2e) = \tau_f(x)(f(\phi_f(x))) = \tau_f(x)(h(\phi_f(x))e)$. This implies that $\tau(x)(2e) = \tau_f(x)(h(\phi_f(x))e)$. As $\tau(x)$ and $\tau_f(x)$ are isometries we get $h(\phi_f(x)) = 2$ and by the choice of h we get $\phi(x) = \phi_f(x)$. Similarly, applying Tf at $\phi(x), \dots, \phi^{n-2}(x)$ we get $\phi^p(x) = \phi_f^p(x)$ for $1 \leq p \leq n-1$. Using the above and the fact that $\tau(x) = \tau_f(x)$ for all $x \in X$ we have $\tau(x) \circ \tau(\phi(x)) \circ \dots \circ \tau(\phi^{n-1}(x)) = \tau_f(x) \circ \tau_f(\phi_f(x)) \circ \dots \circ \tau_f(\phi_f^{n-1}(x)) = I$.

Case(iii): $\phi(x) \neq x$, $\phi^m(x) = x$, m does not divide n and $\phi^s(x) \neq x \forall s$ such that $1 \leq s < m$.

Therefore, \exists integers r and q such that $n = mq + r$, $0 < r < m$. Now, choosing $h \in C(X)$ as in case(ii) and proceeding in the same way we get $\phi^p(x) = \phi_f^p(x)$ for $1 \leq p \leq n-1$. Now, $Tf(\phi^{n-1}(x)) = \tau(\phi^{n-1}(x))(f(\phi^n(x))) = \tau(\phi^{n-1}(x))(h(\phi^n(x))e) = \tau_f(\phi^{n-1}(x))(f(\phi_f(\phi^{n-1}(x)))) = \tau_f(\phi^{n-1}(x))(f(\phi_f(\phi_f^{n-1}(x)))) = \tau_f(\phi^{n-1}(x))(f(\phi_f^n(x))) = \tau_f(\phi^{n-1}(x))(f(x)) =$

$\tau_f(\phi^{n-1}(x))(h(x)e) = \tau_f(\phi^{n-1}(x))(e)$. Thus,
 $\tau(\phi^{n-1}(x))(h(\phi^n(x))e) = \tau_f(\phi^{n-1}(x))(e)$ which implies that $\phi^n(x) = x$. But $\phi^m(x) = x$ implies that $\phi^{mq}(x) = x$ and therefore $x = \phi^n(x) = \phi^{r+mq}(x) = \phi^r(\phi^{mq}(x)) = \phi^r(x)$, a contradiction because $r < m$.

Case(iv): $\phi(x) \neq x, \phi^2(x) \neq x, \dots, \phi^{n-1}(x) \neq x$.

Choose h such that $1 \leq h \leq n$ and $h^{-1}(1) = \{x\}, h^{-1}(2) = \{\phi(x)\}, \dots, h^{-1}(n) = \{\phi^{n-1}(x)\}$. Proceeding the same way as in case (iii) we get $\phi^n(x) = x$ and $\tau(x) \circ \tau(\phi(x)) \circ \dots \circ \tau(\phi^{n-1}(x)) = I$.

□

Our next corollary follows from a result by Jarosz and Rao, see Theorem 7 in [3], and Theorem 2.1.

Corollary 2.2. *Let X be a first countable compact Hausdorff space and E be a uniformly convex Banach space such that $\mathcal{G}(E)$ is algebraically reflexive. Then $\mathcal{G}^n(C(X, E))$ is also algebraically reflexive.*

Corollary 2.3. *Let X and E be as in Corollary 2.2. Then the set of GBP's on $C(X, E)$ is algebraically reflexive.*

Proof. Let the set of all GBP's on $C(X, E)$ be denoted by \mathcal{P} . Let $P \in \overline{\mathcal{P}}^a$. Then for each $f \in C(X, E)$, there exists $P_f \in \mathcal{P}$ such that $Pf = P_f f$. Thus by [2], for each f there exists a homeomorphism ϕ_f of X , $\phi_f^2(x) = x$ for all $x \in X$ and $\tau_f : X \rightarrow \mathcal{G}(E)$ satisfying $\tau_f(x) \circ \tau_f(\phi_f(x)) = I$ such that $Pf(x) = \frac{1}{2}(f(x) + \tau_f(x)(f(\phi_f(x))))$.

Thus for each f , $(2P - I)f(x) = \tau_f(x)(f(\phi_f(x)))$ which implies that $2P - I \in \overline{\mathcal{G}^2(C(X, E))}^a$. The conclusion follows from Corollary 2.2.

□

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REFERENCES

- [1] S. Dutta and T. S. S. R. K. Rao, *Algebraic reflexivity of some subsets of the isometry group*, Linear Algebra Appl. 429 (2008), no. 7, 1522—1527. MR2444339 (2009j:47153).
- [2] F. Botelho and J. E. Jamison, *Generalized bi-circular projections on $C(\Omega, X)$* , Rocky Mountain J. Math. 40 (2010), no. 1, 77—83. MR2607109 (2011c:47039).
- [3] K. Jarosz and T. S. S. R. K. Rao, *Local surjective isometries of function spaces*, Math. Z. 243 (2003), 449—469. MR1970012 (2003m:46036).

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